Therefore, an increase in the axial compressive force T contributes to a rise in the critical external pressure P. The shape of the shell waviness under buckling is represented in Fig. 3.

In conclusion, the author thanks I. I. Vorovich for formulating the problem and discussing the results.

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Translated by M. D. F.

UDC 539.3

INFLUENCE OF THE SIGN OF SHELL CURVATURE ON THE CHARACTER OF THE STATE OF STRESS

PMM Vol. 37, №5, 1973, pp. 952-960
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(Received March 15, 1973)

Problems which turn out to be incorrect in the membrane formulation are investigated. The purpose of this paper is to show that the known anomaly, noted by Vlasov [3], in the behavior of shells of negative curvature and caused by the incorrectness of the formulation of the complete membrane problem for them, is not especially intrinsic property of shells of negative curvature and is observed also in shells of positive curvature, if the complete membrane problem turns out to be incorrect for them. The properties of the stress-strain state are studied as a function of the sign of the middle surface curvature and the manner of edge clamping. The state of stress of the shell is compared with the fundamental state of stress; the edge effect stresses are not taken into account. Two versions of the boundary conditions are considered: one edge of the shell free and the other rigidly clamped (cantilevered shell), and the case when both edges are rigidly clamped.

1. Let us start from the equations and formulas of the bending theory in investigating the state of stress of a thin elastic shell

$$\frac{1}{A}\frac{\partial T_1}{\partial \alpha} + \frac{1}{AB}\frac{\partial B}{\partial \alpha}(T_1 - T_2) + \frac{1}{B}\frac{\partial S}{\partial \beta} + \frac{2}{AB}\frac{\partial A}{\partial \beta}S - \frac{N_1}{R_1} + X = 0 \quad (\alpha\beta) \tag{1.1}$$

$$\frac{T_1}{R_1} + \frac{T_2}{R_2} + \frac{1}{AB}\left[\frac{\partial}{\partial \alpha}(BN_1) + \frac{\partial}{\partial \beta}(AN_2)\right] + Z = 0$$

$$\frac{1}{A}\frac{\partial G_1}{\partial \alpha} + \frac{1}{AB}\frac{\partial B}{\partial \alpha}(G_1 - G_2) - \frac{1}{B}\frac{\partial H}{\partial \beta} - \frac{2}{AB}\frac{\partial A}{\partial \beta}H - N_1 = 0 \quad (\alpha\beta) \tag{1.2}$$

The elasticity relationships are

$$T_1 = \frac{2Eh}{1-\sigma^2}(\varepsilon_1 + \sigma\varepsilon_2) \qquad (\alpha\beta), \qquad S = \frac{Eh}{1+\sigma}\omega \tag{1.3}$$

$$G_1 = -\frac{2Eh^3}{3(1-\sigma^2)}(\varkappa_1 + \sigma\varkappa_2) \qquad (\alpha\beta), \qquad H = \frac{2Eh^3}{3(1+\sigma)}\tau \tag{1.4}$$

The strain-displacement formulas are

$$\varepsilon_1 = \frac{1}{A}\frac{\partial u}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta}v - \frac{w}{R_1} \qquad (\alpha\beta), \qquad \omega = \frac{A}{B}\frac{\partial}{\partial \beta}\frac{u}{A} + \frac{B}{A}\frac{\partial}{\partial \alpha}\frac{v}{B} \tag{1.5}$$

$$\varkappa_1 = \frac{1}{A}\frac{\partial}{\partial \alpha}\left(\frac{1}{A}\frac{\partial w}{\partial \alpha} + \frac{u}{R_1}\right) + \frac{1}{AB}\frac{\partial A}{\partial \beta}\left(\frac{1}{B}\frac{\partial w}{\partial \beta} + \frac{v}{R_2}\right) \qquad (\alpha\beta) \tag{1.6}$$

$$\tau = \frac{1}{A}\frac{\partial}{\partial \alpha}\left(\frac{1}{B}\frac{\partial w}{\partial \beta} + \frac{v}{R_2}\right) - \frac{1}{AB}\frac{\partial A}{\partial \beta}\left(\frac{1}{A}\frac{\partial w}{\partial \alpha} + \frac{u}{R_1}\right) + \frac{1}{R_1}\left(\frac{1}{B}\frac{\partial u}{\partial \beta} - \frac{1}{AB}\frac{\partial B}{\partial \alpha}v\right)$$

The notation from the monograph [1] is used here and it is considered that α and β are dimensionless parameters of an orthogonal coordinate system referred to the lines of curvature of the shell middle surface. The symbol $(\alpha\beta)$ means that another equality can be obtained from the equality which it follows by replacing $(\alpha, 1, A, u, X)$ by $(\beta, 2, B, v, Y)$, respectively.

The complete membrane boundary value problem [2] consisting of the integration of (1.1), (1.3), (1.5) in which $N_1 = N_2 = 0$ will be discussed with just the tangential boundary conditions taken into account and an additional requirement for tangential continuity. This latter means that the quantities u, v, T_1, S should be continuous on an interior line if it coincides with the line $\alpha = \text{const}$, say. Let us note that because of solving the complete membrane problem not only the stress resultants, but also the displacements, are determined. Knowing these latter, the moments can be calculated by means of (1.6), (1.4). The state of stress which is comprised of stresses due to the stress resultants and moments and the edge effect stresses not taken into account here will be called fundamental. As clarified above, the properties of the fundamental state of stress are determined by the character of the complete membrane problem.

The complete membrane problem turns out to be formulated incorrectly sometimes, i.e. there is no correspondence between the kind of equations and the character of the conditions which must be satisfied on the domain boundaries. Vlasov [3] constructed the example of such a problem. He showed that an increased stress and strain appear in hinged shells of negative curvature for certain critical dimensions. At the same time, shells of nonnegative curvature behave in the customary manner under the same boundary conditions. This is associated with the fact that the complete membrane problem in the Vlasov's example does not separate into two Dirichlet problems. Such problems are correct for shells of positive curvature and incorrect for shells of negative curvature. The phenomena noted by Vlasov are in complete agreement with the results of investigations by S. L. Sobolev, who examined the Dirichlet problem in a rectangle for the equations of string vibration [4] and showed that its solution is unstable relative to the rectangle dimensions.

A case of opposite character can be mentioned, when shells of positive curvature have a considerably greater stress than shells of negative curvature. This is observed in cantilever shells, say. The complete membrane problem for such shells separates into two Cauchy problems (static and geometric) for equations of parabolic (for $K = 0$) and elliptic (for $K > 0$) types. Such problems are also called incorrect in mathematical physics and they are poorly investigated [5, 6]. It is known [6] that their solutions are strongly unstable. The instability results in the fact that large reacive forces at the supports are many times greater in the solution of the first static problem than in the problem of the free edge. The degree of such growth is determined by the kind of equation (the growth is according to a power law for a parabolic equation and an exponential law for an elliptic equation) and the magnitude of the domain in which the solution is sought, i.e. the spacing between the clamped and free edges. Futhermore, a second geometric problem is solved. It is of the same type as the static problem. Therefore the displacements on the free edge will be many times greater than those which are "given" on the clamped edge. (The displacements corresponding to a particular solution of the geometric equations (1.5) in which ε_1, ε_2, ω are replaced by the stress resultants T_1, T_2, S by using the elasticity relationships (1.3) are considered given.) These particular solutions will be large because of the instability of the static problem. Eliminating these displacements by using the unstable solution of the geometric problem, we obtain some large displacements, corresponding to the twin growth of the solution described above, on the free edge. The bending strain components (1.6), moments (1.4) and stresses due to the moments can be calculated by means of these displacements. By using the formulas

$$\sigma_T = {}^1/_2 T / h, \quad \sigma_G = {}^3/_2 G / h^2$$

the following estimates can be obtained from (1.1) – (1.6) for the stresses $\sigma_T = 0(h^{-1}p^{-1})$ on the clamped edge; $\sigma_T = 0\ (h^{-1})$, $\sigma_G = 0\ (p^{-2})$ on the free edge. It is assumed here that unit shear stress resultants are given on the free edge and that p is some provisional measure of the growth of the integrals. These are certainly very rough estimates. However, they show that the stresses due to the moments are independent of h. The situation changes when p^{-2} becomes a sufficiently large number, i.e. the double growth of the solution is so large that the solution of the complete membrane problem incorrectly describes the stress-strain state of the shell. It is then necessary to go over to a computation by means of bending theory.

For a shell clamped at two edges the complete membrane problem is a problem of Dirichlet type (with two boundary conditions at each point of the edge). It is correct in the sense mentioned above for shells of positive curvature and is formally incorrect for shells of negative curvature. Moreover, there is a basis to assume that this incorrectness is not essential in this case [2].

It will be shown below by computations that the highest stress originates in cantilevered shells of positive curvature and the lowest in shells of negative curvature. The curvature in clamped shells exerts no essential influence on the character of the state of stress.

In confirmation of the reasoning expressed, the results of the computations are presented for six specific examples of shells obtained by rotating the curves pictured in Fig. 1 around the 00_1 axis. The curves are taken so that their corresponding shells would have the identical altitude and the maximal and minimal radii of the parallels of the sphere and hyperboloid would be equal. The radius of the cylinder is selected so that the error in solving the complete membrane problem would be not too large. For a cantilevered hyperboloid the error in the solution of the complete membrane problem turned out to be negligible. But for a sphere the bending problem had to be solved.

The error has been estimated as follows. As has been noted above, the moments can be found as a result of solving the membrane problem. We evaluate the stress resultants N_1 and N_2, which it had been impossible to find earlier from (1.1) because of their smallness, by using (1.2). Now substituting the known stress resultants N_1, N_2 into (1.1) and integrating, we find the corrections to the shear stress resultants T_1^*, S^*. We consider the greatest of the quantities T_1^* / T_1, S^* / S the error of the computation.

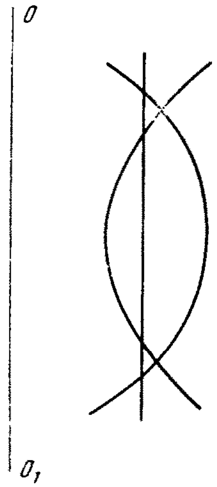

Fig. 1

The load has been selected as $X = Y = 0$, $Z = \cos 2\beta$ in all the cases considered below for rigidly clamped shells, i.e. the solution corresponding to the first term of the Fourier expansion of a load self-equilibrated along the parallels is considered. The loads not self-equilibrated along the parallels are not considered here since this has been investigated in detail in [7].

There is considered to be no surface loads in the examples devoted to the cantilevered shells, and the free edge is considered loaded by the shear stress resultants $T_1 = t(\beta)$, $S = s(\beta)$. Such a choice of load permits better examination of the stress singularities for shells of different curvatures.

2. Let us study the stress-strain state of a cantilevered spherical shell. Let its middle surface be referred to the coordinates α, β related to the geographic coordinates θ, φ by the formulas $\alpha = \ln \mathrm{tg}\,{}^1/_2\theta$, $\beta = \varphi$. The coefficients of the first quadratic form of the surface are $A = B = r\,\mathrm{ch}^{-1}\alpha$ in this coordinate system.

The equality $\zeta = \exp(\alpha + i\beta)$ establishes a one-to-one correspondence between the plane of the complex variable ζ and the sphere given in the α, β coordinate system.

Let us consider the contour $|\zeta| = \rho_2$ subjected to the potential stress resultants $t(\beta)$, $s(\beta)$ and the contour $|\zeta| = \rho_1$ rigidly clamped (ρ_1, $\rho_2 = \mathrm{const}$, $\rho_2 > \rho_1$), i.e. let us consider a cantilever.

It has been shown in [8] that the stress-strain state in a spherical shell is separated into three states: a membrane, a pure bending and a mixed state.

Finding the first two states in the absence of a surface load reduces to constructing two analytic functions $\psi(\zeta)$ and $\chi(\zeta)$, in terms of which the desired stress resultants and moments are expressed directly as

$$-T_1 + iS = r^2 A^{-2}\psi,\quad T_2 = -T_1,\quad N_i = G_i = H = 0 \qquad (i = 1,2)$$
$$-H + iG_1 = r^2 A^{-2}\chi,\quad G_2 = -G_1,\quad T_i = N_i = S = 0$$

In those cases when the variability of the mixed state of stress in one coordinate is considerably greater than in the other, it is more efficient to use the known solution of the simple edge effect instead of seeking the mixed state [1].

The displacements are computed in terms of the functions ψ and χ by means of the formulas [8]

$$\varphi^{(m)} = r \operatorname{ch} \alpha \, (u^{(m)} + i v^{(m)}) = -r^2 \frac{1+\sigma}{2Eh} \int \operatorname{ch}^2 \alpha \bar{\psi} \frac{d\bar{\zeta}}{\bar{\zeta}}$$

$$\varphi^{(b)} = r \operatorname{ch} \alpha \, (u^{(b)} + i v^{(b)}) = r \eta^{-4} \frac{1+\sigma}{2Eh} 3i \int \frac{d\zeta}{\zeta^3} \iint \chi \, d\zeta \, d\zeta$$

Here and henceforth the superscripts (m), (b) denote that the quantities belong, respectively, to the membrane and purely bending states of stress.

We shall seek the solution in the annulus $\rho_1 < |\zeta| < \rho_2$ by giving the desired functions ψ, χ in the form of Laurent series. We write these series for ψ, χ and the displacements thus:

$$\psi = \sum_{m=2} \left(C_{1m} \frac{\zeta^m}{\rho_2^{\,m}} + C_{2m} \frac{\rho_2^{\,m}}{\zeta^m} \right), \qquad i\chi = \sum_{m=2} \left(M_{1m} \frac{\zeta^m}{\rho_2^{\,m}} + M_{2m} \frac{\rho_2^{\,m}}{\zeta_m} \right) \quad (2.1)$$

$$\frac{2Eh}{1+\sigma} \varphi^{(m)} = -r^2 \sum_{m=2} \left(C_{1m} k_{1m} \frac{\zeta^m}{\rho_2^{\,m}} - C_{2m} k_{2m} \frac{\rho_2^{\,m}}{\zeta^m} \right)$$

$$k_{1m} = \frac{1}{4} \left(\frac{\rho^2}{m+1} + \frac{2}{m} + \frac{1}{\rho^2 (m-1)} \right),$$

$$k_{2m} = \frac{1}{4} \left(\frac{\rho^2}{m-1} + \frac{2}{m} + \frac{1}{\rho^2 (m+1)} \right)$$

$$\frac{2Eh}{1+\sigma} \varphi^{(b)} = \eta^{-4} \sum_{m=2} \frac{3r^2}{m(m^2-1)} \left(M_{1m} \frac{\zeta^m}{\rho_2^{\,m}} - M_{2m} \frac{\rho_2^{\,m}}{\zeta^m} \right)$$

Substituting (2.1) and the known expressions for the edge effect quantities into the following boundary conditions

$$T_1 = t_1, \; S - H / r = s, \; G_1 = N_1 - B^{-1} \partial H / \partial \beta = 0 \quad \text{for } |\zeta| = \rho_2$$

$$u = v = w = \gamma = 0 \quad \text{for } |\zeta| = \rho_1$$

we obtain expressions for the coefficients of the series of the desired quantities

$$C_{2m} = \left[-1 - m^2 \operatorname{ch}^2 \alpha_2 + \frac{3\eta^{-4} p^2}{m(m^2-1) k_{1m1}} \right] \frac{t_m}{\Delta \operatorname{ch}^2 \alpha_2} + \quad (2.2)$$

$$\left[m \operatorname{ch} \alpha \operatorname{sh} \alpha - \frac{3\eta^{-4} p^2}{m(m^2-1) \; k_{1m1}} \right] \frac{s_m}{\Delta \operatorname{ch}^2 \alpha_2}$$

$$M_{2m} = \frac{t_m + s_m}{\Delta \operatorname{ch}^2 \alpha_2} + O(\eta^4 p^2), \qquad C_{1m} = \frac{3\eta^{-4} p^2 M_{2m}}{m(m^2-1) k_{1m1}}$$

$$M_{1m} = {}^1/_3 \eta^4 p^2 m (m^2-1) k_{2m1} C_{2m}, \qquad p = \rho_1^{\,m} / \rho_2^{\,m}$$

$$\Delta = -1 - m^2 \operatorname{ch}^2 \alpha_2 - m \operatorname{ch} \alpha_2 \operatorname{sh} \alpha_2 + {}^1/_6 \eta^{-4} p^2 m^{-1} (m^2-1)^{-1} \; k_{1m1}^{-1} + O(p^4)$$

Here t_m, s_m are Fourier coefficients of the expansions of the function $t(\beta)$, $s(\beta)$.

If p is sufficiently small, then the formulas will yield the same result as though the computation had been carried out for an unclamped shell, i.e. the greater the separation between the shell edges, the less a membrane shell will it be. The law according to which the solution grows is exponential (it is a power law in a cylindrical shell).

The membrane condition, or conversely, the pure bending condition can be obtained from (2.2) depending on the parameters therein. Let us just note that if p is sufficiently small, then the membrane condition (for whose derivation $M_{2m} = 0$) is necessary) agrees with the condition obtained in [7] for a spherical shell with one edge which is free. This condition is

$$s_m + t_m = 0 \tag{2.3}$$

Depending on the degree of violation of condition (2.3), the membrane state of stress will be spoiled. However, the practical value of this condition is very small since slight violation of condition (2.3) will cause large bending stresses because of the strong instability in the solution.

3. Let us conduct a detailed computation for shells of negative curvature in the example of a single-sheeted hyperboloid of revolution. Let us refer the middle surface of the hyperboloid to the parallels $z = \text{const}$ and the meridians $\beta = \text{const}$. The following expressions hold in the z, β coordinate system for the coefficients of the first quadratic form A, B and the principal radii of curvature R_1, R_2

$$A = (1 + r'^2)^{1/2},\ B = r,\ R_1 = -A^3 / r'',\ R_2 = Ar$$

where $r = r(z)$ is the equation of the meridian and the prime denotes the derivative with respect to z.

It is known that by using them, the membrane equations for shells of revolution of negative curvature outlined on second order surfaces are reduced to the form [3]

$$\frac{\partial}{\partial\alpha}\left(\frac{r}{A}T_1\right) + \frac{\partial}{\partial\beta}(\lambda r^2 S) = -\lambda r^3 (X + r'Z)$$

$$\frac{\partial}{\partial\beta}\left(\frac{r}{A}T_1\right) + \frac{\partial}{\partial\alpha}(\lambda r^2 S) = -\lambda r^4 A\left(Y - A\frac{\partial Z}{\partial\beta}\right)$$

where

$$r = (az^2 + bz + c)^{1/2} \quad \lambda = 4\,(4ac - b^2)^{-1} \quad \alpha = \operatorname{arc\,tg}\,[{}^1/_2\lambda\,(2az + b)]$$

Replacing the coordinates therein by means of $\xi = \alpha + \beta$, $\eta = \alpha - \beta$ and adding and subtracting these equations in pairs, we ontain a solution of D'Alembert type

$$\frac{r}{A}T_1 + \lambda r^2 S = -\frac{1}{2}\int [\lambda r^3 (X + r'Z) + \lambda^2 r^4 A\,(Y - AZ_{,\beta})]\,d\xi + f_1(\eta)$$

$$\frac{r}{A}T_1 - \lambda r^2 S = -\frac{1}{2}\int [\lambda r^3 (X + r'Z) - \lambda^2 r^4 A\,(Y - AZ_{,\beta})]\,d\eta + f_2(\xi)$$

Here f_1, f_2 are arbitrary functions of integration to be determined from the boundary conditions.

Let us turn first to seeking the state of stress of a cantilivered shell. The boundary conditions for such a construction are

$$u = v = w = \gamma_1 = 0 \quad \text{for} \quad z = z_1; \tag{3.1}$$

$$T_1 = t,\ S = s,\ G_1 = N_1 = 0 \quad \text{for} \quad z = z_2$$

The solutions of the equilibrium equations corresponding to conditions (3.1) are the following, in the absence of surface load:

$$T_1^{(m)} = {}^1/_2 A r^{-1}\,\{A_2 r_2^{-1}\,[t\,(\alpha_2 - \eta) + t\,(\xi - \alpha_2)] + r_2^{-2}\lambda^{-1}\,[s\,(\alpha_2 - \eta) - s\,(\xi - \alpha_2)]\}$$

$$S_1^{(m)} = {}^1/_2 r^{-2}\lambda^{-1}\,\{A_2^{-1} r_2\,[t\,(\alpha_2 - \eta) - t\,(\xi - \alpha_2)] + r_2^{-2}\lambda^{-1}\,[s\,(\alpha_2 - \eta) + s\,(\xi - \alpha_2)]\}$$

$$r_2 = r\,(z_2),\ A_2 = A\,(z_2)$$

The equations for the displacements of the hyperboloid reduce to [3]

$$\frac{\partial}{\partial\alpha}\left(\frac{v}{r}\right)+\frac{\partial}{\partial\beta}(\lambda Au)=\lambda Ar\omega \tag{3.2}$$
$$\frac{\partial}{\partial\beta}\left(\frac{v}{r}\right)+\frac{\partial}{\partial\alpha}(\lambda Au)=\frac{A^2}{rr''}\varepsilon_1+\varepsilon_2$$

The solution of this system is written analogously to the solution for the stress resultants, and hence is not presented.

Let us turn to a specific numerical example, and let us assume

$$t=\cos 2\beta,\ s=\sin 2\beta \quad \text{for } z=z_2,\ r=(1+z^2)^{1/2}$$

Then the desired stress resultants are

$$T_1^{(m)}=Ar^{-1}(C_1\sin 2\alpha-C_2\cos 2\alpha)\cos 2\beta \tag{3.3}$$
$$S^{(m)}=r^{-2}(C_1\cos 2\alpha+C_2\sin 2\alpha)\sin 2\beta$$
$$C_1=r_2^2\cos 2\alpha_2+r_2A_2^{-1}\sin 2\alpha_2,\quad C_2=r_2^2\sin 2\alpha_2-r_2A_2^{-1}\cos 2\alpha_2$$

The graphs of these stress resultants are pictured in Fig. 2 by solid lines.

The tangential conditions for the displacements are: $u=v=0$ for $z=z_1$. The solutions of (3.2) complying with them are representable as follows:

$$2Ehu^{(m)}=\frac{1}{A}\left[\sin 2\alpha\int_{\alpha_2}^{\alpha}(h_1\sin 2\alpha-h_2\cos 2\alpha)\,d\alpha+\right. \tag{3.4}$$
$$\left.\cos 2\alpha\int_{\alpha_2}^{\alpha}(h_1\cos 2\alpha+h_2\sin 2\alpha)\,d\alpha\right]\cos 2\beta$$
$$2Ehv^{(m)}=r\left[\sin 2\alpha\int_{\alpha_2}^{\alpha}(h_2\sin 2\alpha+h_1\cos 2\alpha)\,d\alpha+\right.$$
$$\left.\cos 2\alpha\int_{\alpha_2}^{\alpha}(h_2\cos 2\alpha-h_1\sin 2\alpha)\,d\alpha\right]\sin 2\beta$$
$$h_1=[2-2\sigma+4z^4(2z^2+1)^{-1}]T_1^{(m)},\quad h_2=2(1+\sigma)(2z^2+1)^{1/2}S^{(m)}$$
$$w^{(m)}=2Av^{(m)}+zr^{-1}u^{(m)}-(1-\sigma Ar)T_1^{(m)}(2Eh)^{-1}$$
$$u^{(b)}=A^{-1}(D_1\cos 2\alpha+D_2\sin 2\alpha)\cos 2\beta$$
$$v^{(b)}=r(D_1\sin 2\alpha-D_2\cos 2\alpha)\sin 2\beta$$
$$w^{(b)}=2Av^{(b)}+zr^{-1}u^{(b)}$$
$$D_1=-v^{(m)}(\alpha_1)r_1^{-1}\sin 2\alpha_1-A_1u^{(m)}(\alpha_1)\cos 2\alpha_1$$
$$D_2=v^{(m)}(\alpha_1)r_1^{-1}\cos 2\alpha_1-A_1u^{(m)}(\alpha_1)\sin 2\alpha_1$$

It is interesting to note that although the solution (3.3) for the hyperboliod is expressed in terms of trigonometric functions, it is not oscillatory. This is because α varies between $-1/2\,\pi$ and $+1/2\pi$ when z varies between $-\infty$ and ∞, and therefore, the solutions (3.3) in a hyperboloid of no matter what length can change sign no more than twice. This change of sign will hence occur more closely to the point $z = 0$ the greater the λ, i.e. the greater the curvature of the "neck". Membrane theory can here become inapplicable because of great variability in the state of stress.

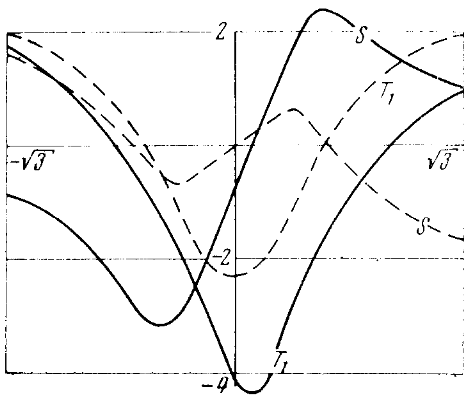

Fig. 2

The computation of a hyperboloid clamped rigidly along two contours was carried out numerically for the following load:

$$X = Y = 0, \; Z = \cos 2\beta$$

The boundary conditions of rigid clamping are written as

$$u = v = w = \gamma_1 = 0 \text{ for } z = z_1, \; z = z_2 \qquad (3.5)$$

Expressions for the stress resultants (3.3) and displacements (3.4) were used in the calculations. The process of complying with the boundary conditions (3.5) is well known. The arbitrary constants C_2, D_2 are determined from the first two boundary conditions ($C_1 = D_1 = 0$ follows from the symmetry of the structure), and are used to evaluate the stress resultants and the displacements. Graphs of the stress resultants for a hyperboloid clamped on two contours are pictured by dashed lines in Fig. 2. The error in the computation is estimated to be

$$T_1^* \,/\, T_1 = O\,(10^{-4})$$

4. The process of analyzing a cylindrical shell is not described here since it has been elucidated in detail in [1]. The stresses for all shells have been computed for $h = 0.01$.

The maximum stresses in cantilevered shells are: (a) in a cylindrical shell $\sigma_T=200$ at the support, $\sigma_G = 50$ at the free edge; (b) in a spherical shell (it is assumed that $t_2 = s_2 = 1$) $\sigma_T = 19$, $\sigma_G = 765$ at the free edge (the stresses at the support are negligible); (c) in a hyperbolic shell $\sigma_T = 210$, $\sigma_G = 40$ at the neck of the hyperboloid $\sigma_G = 6.6$ at the free edge.

The stresses in rigidly clamped shells are : (a) in a cylindrical shell $\sigma_T = 70$, $\sigma_G = 0.85$ at the middle of the shell ; (b) in a spherical shell $\sigma_T = 150$ at the equator, $\sigma_G = 7.4$ at the edge ; (c) in a hyperbolic shell $\sigma_T = 110$, $\sigma_G = 27$ at the neck.

These numbers verify the general reasoning. As expected, the bending stresses turn out to be greatest in a spherical cantilever, and smallest in the hyperbolic. The stresses in rigidly clamped shells differ considerably less.

Assurance of membrane behavior as a result of the clamping of one edge is successfully achieved in a hyperboloid to a large extent, and in a cylinder to a lesser extent. Clamping plays a part for a spherical contilever only in the case of a sufficiently narrow spherical belt. This result is because the Cauchy's problem for equations of elliptic type must be solved in the problem concerning the state of stress of a spherical cantilever. The strong instability of such a solution indeed resulted in the appearance of large displacements at the free edge which caused large transverse stress resultants and moments.

5. Let us analyze the statics of the problems considered in order to explain the physical meaning of the appearance of bending stresses in shells.

Let us cut a panel $\beta \in (^{-1}/_4\pi,\ ^1/_4\pi)$, $\alpha \in (\alpha_1, \alpha_2)$ (Fig. 3) from a circular cylindrical shell carrying the edge load

$$t = t_2\cos 2\beta,\ s = s_2\sin 2\beta \tag{5.1}$$

Reactive forces in the clamping and internal forces applied to the rectilinear edges of the pannel should equalize the edge load. The former should be [1]

$$T_2 = T_{22}\cos 2\beta,\ G_2 = G_{22}\cos 2\beta,\ S = S_{22}\sin 2\beta,\ N_2 = N_{22}\sin 2\beta \tag{5.2}$$

and hence on the rectilinear edge

$$T_2 = G_2 = 0 \tag{5.3}$$

Under the action of the edge forces the panel tends to rotate counter-clockwise around the line OO_1. The forces N_2 and the reaction R will hinder this. Let us assume that $N_2 = 0$. Then the behavior of the panel will not differ from the behavior of an analogous beam whose rectilinear edges experience displacements easily computed by strength of materials formulas. If the panel pictured in Fig. 3 tends to rotate counter-clockwise then its vicinity in the shell will tend to rotate clockwise. But there is no displacement w at the rectilinear edges in the shell during the deformation of the panel since $w = w_2\cos 2\beta$. Therefore the forces N_2 straighten the beam-panel in conformity with the requirements for their joint behavior in the shell. The reason why the forces N_2 are large even in a short cantilever shell now becomes evident. The forces N_2 become the principal forces in a sufficiently long shell and the reaction R is secondary since the reaction R does not influence the deflection. And it is found that the edge effect in a long shell is equalized by forces N_2 which should damp out in conformity whith the geometric scheme of shell behavior considered above, and to damp out more rapidly the thicker the shell.

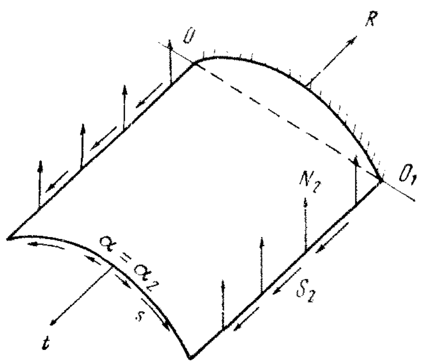

Fig. 3

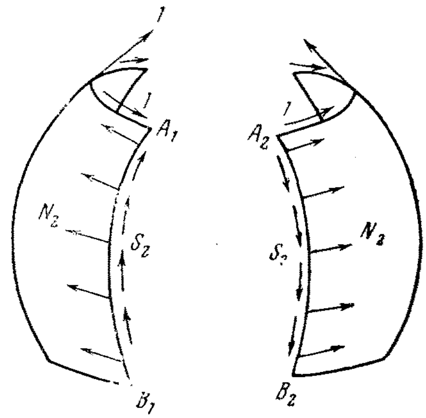

Fig. 4

Let us cut a section $\beta \in (-^1/_4\ \pi,\ ^1/_4\pi)$, $|\ \xi\ | \in (\rho_1, \rho_2)$ out of a spherical shell loaded by edge forces (5.1) in the tangential plane. Two such sections, loaded differently by internal forces, are pictured in Fig. 4. For the sections cut out by the method mentioned (5.2), (5.3) are true and this is taken into account in loading the sides $\beta_1 = -\ ^1/_4\pi$, $\beta_2 = {}^1/_4\pi$. Edge forces whose Fourier coefficients are $t_2 = s_2 = 1$ in (5.1) are applied to the section on the left in Fig. 4, while $t_2 = -s_2 = 1$ for the right section of the shell. We take the direction of the force S_2 at the corners A_1 and A_2 in conformity with the rule of signs. Furthermore, the signs of the stress resultants do not change during motion from point A_1 to B_1 and from A_2 to B_2 (this follows from the solution which is known). In the first case the moments of all the tangential (internal and external) forces are added with the same sign, and in the second case with different signs. It is clear that the forces N_2 are perfectly necessary in the first case, and perhaps are not in the second. (The solution shows that $N_2 = 0$.)

No part which cannot be equalized without taking account of the transverse forces and moments, cannot be found for a shell of negative curvature. Moreover, the moments due to the forces S_1 and S_2 acting on the part of the shell conveniently cut out as had been done above, are opposite in sign while they are identical in sign in shells of positive curvature. Therefore, there is less foundation to expect bending stresses to appear in a shell of negative curvature than in a shell of positive curvature, which is in agreement with the results of computations.

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Translated by M. D. F.